Technical Notes

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Bordering Algorithm for Solution of Boundary-Layer Equations in Inverse Mode

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Introduction

It is well known that the solution of boundary-layer equations can be achieved in the form of a tridiagonal matrix.[1] In the case of inverse calculations, such as those associated with separated flows,[1,2] the external velocity u_e becomes part of the solution, and although the tridiagonal matrix can still be used, there are consequences, which include an increase in computing requirements. The following section describes an algorithm, based on that of Ref. 3, which allows the formulation of the inverse problem to conform to that where the external velocity is prescribed. The minor changes required within the framework of the box scheme[1] are described.

Bordering Algorithm

It is convenient to solve the two-dimensional steady incompressible laminar boundary-layer equations in the form

$$f''' + \tfrac{1}{2}ff'' + xw\frac{dw}{dx} = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{1}$$

$$\eta = 0, \quad f = f' = 0, \qquad \eta = \eta_e, \quad f' = w \tag{2}$$

where

$$w = u_e/u_0, \qquad \eta = \sqrt{(u_0/\nu x)}y, \qquad \psi = \sqrt{u_0 \nu x}f(x, \eta)$$

Using Keller's box method,[1,4] Eqs. (1) and (2) are expressed as a system of three first-order equations:

$$f' = u, \qquad u' = v \tag{3a}$$

$$v' + \tfrac{1}{2}fv + xw\frac{dw}{dx} = x\left(u\frac{\partial u}{\partial x} - v\frac{\partial f}{\partial x}\right) \tag{3b}$$

$$\eta = 0, \quad f = u = 0; \qquad \eta = \eta_e, \quad u = w \tag{4}$$

Next, on a finite-difference net, denoted by $1 \le n \le N$, $1 \le j \le J$,

$$x_0 = 0, \qquad x_n = x_{n-1} + k_{n-1}, \qquad n = 1, 2, \ldots, N$$

$$\eta_0 = 0, \qquad \eta_j = \eta_{j-1} + h_{j-1}, \qquad j = 1, 2, \ldots, J; \quad \eta_J = \eta_e$$

the finite-difference approximations to Eqs. (3) and (4) are written and linearized with Newton's method, and the resulting system, with the coefficients $(s_k)_j$ $(k = 1\text{–}7)$ and $(r_k)_j$ $(k = 1\text{–}3)$ given in Ref. 1, is

$$\delta f_j - \delta f_{j-1} - (h_{j-1}/2)(\delta u_j + \delta u_{j-1}) = (r_1)_j \tag{5a}$$

$$\delta u_j - \delta u_{j-1} - (h_{j-1}/2)(\delta v_j + \delta v_{j-1}) = (r_3)_{j-1} \tag{5b}$$

$$(s_1)_j\delta v_j + (s_2)_j\delta v_{j-1} + (s_3)_j\delta f_j + (s_4)_j\delta f_{j-1} + (s_5)_j\delta u_j + (s_6)_j\delta u_{j-1} + (s_7)_j\delta w = (r_2)_j \tag{5c}$$

$$\delta f_0 = 0, \qquad \delta u_0 = 0, \qquad \delta u_J = \delta w \tag{6}$$

In cases where δw must be determined as part of the solution, interaction between inviscid and boundary-layer equations may be achieved by using the Hilbert integral, as suggested by Veldman,[5] or by specifying the displacement thickness, as by Carter and Wornum.[6] The interaction formula can be expressed as

$$\gamma_1\delta f_J + \gamma_2\delta w = (r_3)_J \tag{7}$$

where γ_1, γ_2, and $(r_3)_J$ are coefficients that depend on the particular formulation.

With the introduction of appropriate vectors and matrices, the preceding linear system can be written in a matrix-vector form[1] as

$$A\,\underset{\approx}{\delta} = \underset{\approx}{r} \tag{8}$$

$$A = \left|\begin{array}{c|c} a & D \\ \hline E^T & A_J \end{array}\right|, \qquad \underset{\approx}{\delta} = \left|\begin{array}{c} \underset{\sim}{\Delta} \\ \underset{\sim}{\delta}_J \end{array}\right|, \qquad \underset{\approx}{r} = \left|\begin{array}{c} \underset{\sim}{R} \\ \underset{\sim}{r}_J \end{array}\right| \tag{9}$$

$$a = \left|\begin{array}{cccccc} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & B_j & A_j & C_j & \\ & & & \cdot & \cdot & \cdot \\ & & & B_{J-2} & A_{J-2} & C_{J-2} \\ & & & & B_{J-1} & A_{J-1} \end{array}\right| \tag{10}$$

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$$D = \begin{vmatrix} D_0 \\ \vdots \\ D_j \\ \vdots \\ J_{J-1} \end{vmatrix}, \quad E = \begin{vmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ B_J \end{vmatrix}, \quad \underset{\sim}{\Delta} = \begin{vmatrix} \underset{\sim}{\delta}_0 \\ \vdots \\ \underset{\sim}{\delta}_j \\ \vdots \\ \underset{\sim}{\delta}_{J-1} \end{vmatrix}$$

$$\underset{\sim}{\delta}_j = \begin{vmatrix} \delta f_j \\ \delta u_j \\ \delta v_j \end{vmatrix}, \quad 0 \le j \le J$$

$$\underset{\sim}{R} = \begin{vmatrix} \underset{\sim}{r}_0 \\ \vdots \\ \underset{\sim}{r}_j \\ \vdots \\ \underset{\sim}{r}_{J\text{-}1} \end{vmatrix}, \quad \underset{\sim}{r}_0 = \begin{vmatrix} 0 \\ 0 \\ (r_3)_0 \end{vmatrix}, \quad \underset{\sim}{r}_j = \begin{vmatrix} (r_1)_j \\ (r_2)_j \\ (r_3)_j \end{vmatrix}$$

$$1 \le j \le J \tag{11a}$$

In Eq. (10), A_j, B_j, C_j, and D_j are 3×3 block matrices given by

$$A_0 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -h_0/2 \end{vmatrix}$$

$$A_j = \begin{vmatrix} 1 & -h_{j-1}/2 & 0 \\ (s_3)_j & (s_5)_j & (s_1)_j \\ 0 & -1 & -h_j/2 \end{vmatrix}$$

$$1 \le j \le J-1$$

$$A_J = \begin{vmatrix} 1 & -h_{J-1}/2 & 0 \\ (s_3)_J & (s_5)_J + (s_7)_J & (s_1)_J \\ \gamma_1 & \gamma_2 & 0 \end{vmatrix}$$

$$B_j = \begin{vmatrix} -1 & -h_{j-1}/2 & 0 \\ (s_4)_j & (s_6)_j & (s_2)_j \\ 0 & 0 & 0 \end{vmatrix}$$

$$1 \le j \le J$$

$$C_j = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -h_j/2 \end{vmatrix}, \quad 0 \le j \le J-2 \tag{11b}$$

$$D_0 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad D_j = \begin{vmatrix} 0 & 0 & 0 \\ 0 & (s_7)_j & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$1 \le j \le J-2 \tag{11c}$$

$$D_{J-1} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & (s_7)_{J-1} & 0 \\ 0 & 1 & -h_J/2 \end{vmatrix}$$

For convenience, we write the D matrix as

$$D = [\underset{\sim}{D}_1, \ \underset{\sim}{D}_2, \ \underset{\sim}{D}_3] \tag{12}$$

where

$$\underset{\sim}{D}_1 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \underset{\sim}{D}_2 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (s_7)_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (s_7)_j \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (s_7)_{J-1} \\ 1 \end{vmatrix}, \quad \underset{\sim}{D}_3 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ -h_J/2 \end{vmatrix} \tag{13}$$

The term $(s_7)_j \delta w$ in Eq. (5c) causes the coefficient marix A to have a form other than tridiagonal, as indicated in Eq. (9), and this prevents the use of the efficient block-elimination method discussed in Ref. 1. A consequence was the development and use of the Mechul function approach,[1] where w_j was an extra unknown and the coefficient matrix A became tridiagonal with 4×4 blocks rather than the more convenient 3×3 arrangement normally associated with standard two-dimensional boundary layers. An immediate consequence was reduction of the efficiency of the numerical method and, as the number of equations increases to represent compressibility, heat transfer, three-dimensional flows, and transport turbulence models, this deviation from the tridiagonal arrangement makes the solution of the linear system increasingly more difficult.

To overcome this deficiency and to return to a 3×3 tridiagonal arrangement, we use the bordering algorithm in which we write Eq. (8) as

$$a \underset{\sim}{\Delta} + D \underset{\sim}{\delta}_J = \underset{\sim}{R} \tag{14a}$$

$$E^T \underset{\sim}{\Delta} + A_J \underset{\sim}{\delta}_J = \underset{\sim}{r}_J \tag{14b}$$

If we introduce F such that

$$aF = D \tag{15}$$

then by substituting Eq. (15) into Eq. (14a), we get

$$a \underset{\sim}{w} = \underset{\sim}{R} \tag{16}$$

where

$$\underset{\sim}{w} = \underset{\sim}{\Delta} + F \underset{\sim}{\delta}_J \tag{17}$$

With Eq. (17), Eq. (14b) can be written as

$$(A_J - E^T F) \underset{\sim}{\delta}_J = \underset{\sim}{r}_J - E^T \underset{\sim}{w} \tag{18}$$

and the solution of Eq. (8) is obtained in the following three steps:

1) Solve Eqs. (15) and (16)

$$a \underset{\sim}{w} = \underset{\sim}{R}, \qquad aF = D \tag{19}$$

where $\underset{\sim}{w}$ and $\underset{\sim}{R}$ are vectors and F and D are matrices. Note that, since the coefficient matrix a has a block-tridiagonal structure, the solution of Eq. (19) can be obtained efficiently with the block-elimination method. With $F = [\underset{\sim}{F}_1, \ \underset{\sim}{F}_2, \ \underset{\sim}{F}_3]$, the second part of Eq. (19) can be written as vector equations with the definition of Eq. (12). Note that $\underset{\sim}{D}_1$ is zero and that we need to solve only two vector equations:

$$a\,\underset{\sim}{F}_2 = \underset{\sim}{D}_2, \qquad a\,\underset{\sim}{F}_3 = \underset{\sim}{D}_3 \tag{20}$$

2) With $\underset{\sim}{w}$ and F known from step 1, compute

$$A'_J = A_J - E^T F \tag{21}$$

and

$$\underset{\sim}{r}'_J = \underset{\sim}{r}_J - E^T \underset{\sim}{w} \tag{22}$$

and then solve

$$A'_J\,\underset{\sim}{\delta}_J = \underset{\sim}{r}'_J \tag{23}$$

which can be done easily since A'_J is a 3×3 matrix, and it is simple to solve a 3×3 system to obtain $\underset{\sim}{\delta}_J$.

3) Now compute

$$\underset{\sim}{\Delta} = \underset{\sim}{w} - F\,\underset{\sim}{\delta}_J \tag{24}$$

The above-described algorithm has been applied to problems where the Mechul function approach had been required previously. The results have shown it to be more convenient to use with considerable savings of computer time. It can be expected that these savings will be even greater for problems involving a larger number of differential equations.

Acknowledgments

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References

[1]Bradshaw, P., Cebeci, T., and Whitelaw, J. H., *Engineering Calculation Methods for Turbulent Flows,* Academic, New York, 1981.

[2]Klineberg, J. M., and Steger, J. L., "On Laminar Boundary-Layer Separation," AIAA Paper 74-94, 1974.

[3]Keller, H. B., "The Bordering Algorithm and Path Following Near Singular Points of Higher Nullity," *SIAM Journal on Scientific and Statistical Computing,* Vol. 4, No. 4, 1983, pp. 573-582.

[4]Keller, H. B., "A New Difference Scheme for Parabolic Problems," *Numerical Solution of Partial-Differential Equations,* Vol. II, edited by J. Bramble, Academic, New York, 1970.

[5]Veldman, A. E. P., "New Quasi-Simultaneous Method to Calculate Interacting Boundary Layers," *AIAA Journal,* Vol. 19, 1981, p. 769.

[6]Carter, J. E., and Wornum, S. F., "Solutions for Incompressible Separated Boundary Layers Including Viscous-Inviscid Interaction," NASA SP-347, 1975.

Stratified Flow Around an Axisymmetric Body at Small Angle of Attack

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Background

THE characterization of the internal wavefield produced by the wake of a vehicle moving in a stratified fluid is a problem in statistics. The recognition that the wavefield contains a prominent random component, generated by wake turbulence, led to the application of experimental methods based on ensemble averaging so that both the stochastic radiation and the mean properties of the wavefield could be described. For convenience of study, the mean properties can be thought to be built from a number of separate sources: body displacement, wake collapse (i.e., a sudden flattening of the wake by hydrostatic forces), propeller swirl, and hydrodynamic lift. In reality, all are coupled in a complicated manner, and shuffle in relative importance as operating conditions change.

The first published results from the ensemble testing program appeared in Ref. 1. These results not only confirmed the importance of the turbulence-generated random internal wavefield, but among other things, they revealed the existence of a sizable propeller-induced wavefield. This particular component was isolated by reducing the mean wavefield into even and odd parts (with respect to the axis of travel) and associating the antisymmetric part with propeller influences. Here we will be interested in the average symmetric wavefield only, so these influences must be removed. Nominally, all of the tests reported in Ref. 1 were conducted at zero angle of attack, and, in the present context, they serve as a baseline for studying the added effects of hydrodynamic lift.

When an axisymmetric body moves at an angle of attack, it generates lift even in the absence of wings or control surfaces. At large angle of attack (say $\alpha > 5$ deg), the resulting circulation can be considered to be deposited into the wake in the form of a turbulent vortex pair. This description is unlikely to be completely adequate at the small incidence angles of interest here—the vorticity being more distributed. However, regardless of the details, net vertical momentum is imparted to the wake, altering both its dynamic and kinematic development, and, in particular, its capacity to generate internal waves.

Existing analytical procedures for estimating the lift coefficient (C_L) of a streamlined, axisymmetric body as a function of its angle of attack cannot be criticized for being overly refined. At one extreme, slender body theory predicts the lift coefficient to be zero, which obviously is of no help in formulating a forcing mechanism for wave theory. The state of the subject is summarized in Ref. 2, where one is discouraged to find that estimates of C_L can differ by more than a factor of 2. This circumstance naturally presents difficulties in comparing theory and experiment, but we will see that choosing an intermediate value of C_L appears to produce reasonable results.

An approximate theoretical description of the kinematics of a turbulent mass with vertical momentum, rising in a stratified medium, has been given by Tulin and Shwartz[3] using similarity arguments. Although this theory is rudimentary and somewhat removed from the present flow situation, it has the virtue of being simple and, for this reason, was the first candidate examined.

Results and Discussion

Internal Wave Experiments

The experiments were conducted in the 30-ft towing basin at the Applied Physics Laboratory Hydrodynamics Research Laboratory. The testing procedure are described fully in Ref. 1, where it is noted that extraordinary care was needed to assure that the test conditions were very nearly identical for each member of the ensemble. Measurements in the (deterministic) potential flow region indicated that the standard error in vertical displacement was only 0.1% of the vehicle diameter. For the present tests, the single-electrode conductivity probes, used for measuring vertical displacements, were arranged as shown in Fig. 1. This figure also defines the coordinate system.

The test vehicle (12:1 fineness ratio) was operated at an average angle of attack of -2.0 deg (nose down). The present ensemble consisted of 20 individual runs at an internal Froude number (F) of 52, where $F = 2\pi V/ND$ and $N^2 = -g/\rho(d\rho/dz)$(N is the Brunt-Vaisala frequency). The ensemble-averaged

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